CHEN–RUAN COHOMOLOGY OF SOME MODULI SPACES

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Abstract. Let X be a compact connected Riemann surface of genus at least two. We compute the Chen–Ruan cohomology ring of the moduli space of stable $\mathrm{PSL}(2,\mathbb{C})$–bundles of nontrivial second Stiefel–Whitney class over X.

1. Introduction

The Chen–Ruan cohomology ring of an orbifold, introduced in [CR1], is the degree zero part of the small quantum cohomology ring of the orbifold constructed by the same authors [CR2] from the moduli space of orbifold morphisms of orbifold spheres into the orbifold. It contains the usual cohomology ring of the orbifold as a subring. The cohomology groups associated to it were known earlier in the literature as orbifold cohomology groups, primarily due to the work of string theorists, for orbifolds that are quotients of a manifold by action of a finite group. For a large class of compact orbifolds, namely quotient of a smooth manifold by foliated action of a compact Lie group, the Chen–Ruan cohomology group (with $\mathbb{Z}/2\mathbb{Z}$ grading) is isomorphic to equivariant K–theory via an equivariant Chern character map (see [AR]). If the orbifold has an algebraic structure, then the Betti numbers of Chen–Ruan cohomology are invariant under crepant resolutions (see [LP] and [Ya]), which underscores their importance in Calabi–Yau geometry. The ring structure behaves more subtly under resolution, but is conjectured by Ruan (see [Ru]) to be isomorphic to the cohomology ring of a smooth crepant resolution if both the orbifold and the resolution are hyper–Kähler. This has been proved in the local case by Ginzburg–Kaledin [GK], and for the symmetric product of a projective $K3$ surface by Fantechi–Göttsche [FG], and Uribe [Ur]. Our aim here is to compute the Chen–Ruan cohomology ring of a certain type of moduli spaces of vector bundles which we describe next.

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. Fix a holomorphic line bundle ξ over X such that

$$\mathrm{degree}(\xi) = 1 .$$

Let $\mathcal{M}_\xi$ denote the moduli space that parametrizes the isomorphism classes of stable vector bundles E over X with $\mathrm{rank}(E) = 2$ and $\det E := \bigwedge^2 E = \xi$. This moduli space $\mathcal{M}_\xi$ is an irreducible complex projective manifold of complex dimension $3g-3$.

Let

$$\Gamma := \mathrm{Pic}^0(X)_2 \subset \mathrm{Pic}^0(X) \tag{1.1}$$

2000 *Mathematics Subject Classification.* 14D20, 14F05.

Key words and phrases. Orbifold, Chen–Ruan cohomology, vector bundle, moduli space.

be the group of line bundles L over X satisfying the condition that $L\bigotimes L$ is holomorphically trivial. So Γ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2g}$. The group Γ acts on $\mathcal{M}_\xi$ as follows.

Take any $L\in\Gamma$. Let

$$\phi_L : \mathcal{M}_\xi \longrightarrow \mathcal{M}_\xi \tag{1.2}$$

be the holomorphic automorphism defined by $E\longmapsto E\bigotimes L$. Let

$$\phi : \Gamma \longrightarrow \mathrm{Aut}(\mathcal{M}_\xi) \tag{1.3}$$

be the homomorphism defined by $L\longmapsto \phi_L$.

The quotient space $\mathcal{M}_\xi/\Gamma$ is the moduli space of stable $\mathrm{PSL}(2,\mathbb{C})$–bundles E over X such that the second Stiefel–Whitney class

$$w_2(E)\in H^2(X,\,\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

is nonzero. For any $E\in\mathcal{M}_\xi$, the corresponding $\mathrm{PSL}(2,\mathbb{C})$–bundle is the one defined by the projective bundle $\mathbb{P}(E)$ associated to E.

We compute the Chen–Ruan cohomology ring of the orbifold $\mathcal{M}_\xi/\Gamma$.

For each element $L\in\Gamma$, let

$$\mathcal{S}(L)\subset\mathcal{M}_\xi \tag{1.4}$$

be the smooth subvariety that is fixed pointwise by the automorphism ϕ_L constructed in (1.2). Since Γ is abelian, the action of Γ on $\mathcal{M}_\xi$ preserves $\mathcal{S}(L)$. The Chen–Ruan cohomology group $H^*_{CR}(\mathcal{M}_\xi/\Gamma,\,\mathbb{Q})$ is defined to be

$$H^*_{CR}(\mathcal{M}_\xi/\Gamma,\,\mathbb{Q}) = H^*(\mathcal{M}_\xi/\Gamma,\mathbb{Q})\bigoplus\left(\bigoplus_{L\in\Gamma\setminus\{\mathcal{O}_X\}} H^{*-2\iota(L)}(\mathcal{S}(L)/\Gamma,\,\mathbb{Q})\right). \tag{1.5}$$

Note that the first summand is in fact the contribution of $\mathcal{S}(L)/\Gamma$ corresponding to $L=\mathcal{O}_X$. The degree shift $2\iota(L)$ is a locally constant function of the action of L on $T_E\mathcal{M}_\xi$, where $E\in\mathcal{S}(L)$. This $\iota(L)$ is determined by the eigenvalues along with their multiplicities, of the differential $d\phi_L$. The ring structure of Chen–Ruan cohomology is defined via a three point correlation function (see (6.21)) and involves virtual classes or obstruction bundles as in Gromov–Witten theory. In our situation, it suffices to know the ranks of these obstruction bundles.

2. A characterization of the fixed point sets

Take any holomorphic line bundle L over X such that $L\bigotimes L$ is holomorphically trivial. Fix a nonzero holomorphic section

$$s\in H^0(X,\,L^{\otimes 2}).$$

Since $L^{\bigotimes 2}$ is trivial, the section s does not vanish at any point of X. Let

$$Y_L := \{z\in L \mid z^{\otimes 2}\in \mathrm{image}(s)\} \tag{2.1}$$

be the complex projective curve in the total space of L. Let

$$\gamma_L : Y_L \longrightarrow X \tag{2.2}$$

be the restriction of the natural projection $L \longrightarrow X$. Consider the action of the multiplicative group $\mathbb{C}^*$ on the total space of L. The action of the subgroup

$$C_2 := \{z \in \mathbb{C} \mid z^2 = 1\} \subset \mathbb{C}^* \tag{2.3}$$

preserves the curve Y_L in (2.1). Consequently, Y_L is a principal C_2–bundle over X. In other words, the projection γ_L makes Y_L an unramified Galois covering of X with Galois group

$$\mathrm{Gal}(\gamma_L) = C_2 . \tag{2.4}$$

Since any two nonzero sections of L differ by multiplication with a nonzero constant scalar, the isomorphism class of the covering γ_L does not depend on the choice of the section s. (See [BNR, p. 173, Example 3.4].)

Let

$$\sigma : Y_L \longrightarrow Y_L \subset L \tag{2.5}$$

be the automorphism defined by multiplication with -1.

If the line bundle L is nontrivial, then Y_L is connected. In that case the genus of Y_L is $2g-1$. If L is the trivial line bundle, then Y_L is the disjoint union of two copies of X, and σ in (2.5) simply interchanges the two components.

Lemma 2.1. *Let L be a nontrivial holomorphic line bundle over X of order two. Take a holomorphic line bundle η over Y_L (see (2.2)) of degree one. Then the direct image*

$$\gamma_{L*}\eta \longrightarrow X$$

is a stable vector bundle over X of rank two and degree one.

Proof. Since the covering γ_L is unramified,

$$\mathrm{degree}(\gamma_{L*}\eta) = \mathrm{degree}(\eta) = 1 .$$

We note that

$$\gamma_L^*\gamma_{L*}\eta = \eta \bigoplus \sigma^*\eta , \tag{2.6}$$

where σ is defined in (2.5). Since $\mathrm{degree}(\sigma^*\eta) = \mathrm{degree}(\eta)$, the right–hand side in (2.6) is a polystable vector bundle on Y_L. Consequently, the vector bundle $\gamma_{L*}\eta$ is polystable. Now we conclude that $\gamma_{L*}\eta$ is stable because $\mathrm{rank}(\gamma_{L*}\eta)$ is coprime to $\mathrm{degree}(\gamma_{L*}\eta)$. □

Fix a holomorphic line bundle ξ over X of degree one. As before, by $\mathcal{M}_\xi$ we denote the moduli space of stable vector bundles of rank two over X with $\bigwedge^2 E = \xi$.

Proposition 2.2. *Let $L \in \Gamma \setminus \{\mathcal{O}_X\}$ be a line bundle over X of order two.*

1. *Take any $\eta \in \mathrm{Pic}^1(Y_L)$, where Y_L is constructed in (2.2), such that the line bundle $\bigwedge^2 \gamma_{L*}\eta$ is isomorphic to ξ. Then $\gamma_{L*}\eta \longrightarrow X$ is a fixed point of the automorphism ϕ_L constructed in (1.2).*
2. *Let $E \in \mathcal{M}_\xi$ be such that $\phi_L(E) = E$, where ϕ_L is the map in (1.2). Then there is a holomorphic line bundle η over Y_L (see (2.2)) such that the direct image $\gamma_{L*}\eta$ is isomorphic to E.*

(3) *Let η_1 and η_2 be holomorphic line bundles over Y_L of degree one. Then the direct image $\gamma_{L*}\eta_1$ is isomorphic to $\gamma_{L*}\eta_2$ if and only if there is a unique element*
$$\tau \,\in\, \mathrm{Gal}(\gamma_L) \,=\, \mathbb{Z}/2\mathbb{Z}$$
of the Galois group for γ_L such that $\eta_1 \,=\, \tau^*\eta_2$.

Proof. From Lemma 2.1 we know that for any $\eta' \,\in\, \mathrm{Pic}^1(Y_L)$, the direct image $\gamma_{L*}\eta'$ is stable. Hence $\gamma_{L*}\eta$ in the first part of the proposition lies in $\mathcal{M}_\xi$. The pull back of any line bundle L_1 to the complement of the zero section of L_1 has a canonical trivialization. In particular, the pull back γ_L^*L has a canonical trivialization. Therefore, we have a natural isomorphism
$$h \,:\, \eta \,=\, \eta \bigotimes_{\mathcal{O}_X} \mathcal{O}_X \,\longrightarrow\, \eta \bigotimes \gamma_L^* L$$
which is obtained by tensoring Id_η with the homomorphism $\mathcal{O}_X \,\longrightarrow\, \gamma_L^*L$ defining the trivialization of γ_L^*L. The above isomorphism h induces an isomorphism
$$\gamma_{L*}h \,:\, \gamma_{L*}\eta \,\longrightarrow\, \gamma_{L*}(\eta \bigotimes \gamma_L^*L) \,=\, (\gamma_{L*}\eta) \bigotimes L \tag{2.7}$$
with the isomorphism $\gamma_{L*}(\eta \otimes \gamma_L^*L) \,=\, (\gamma_{L*}\eta) \otimes L$ being given by the projection formula. Hence $\gamma_{L*}\eta$ is a fixed point of the automorphism ϕ_L in (1.2). This proves statement (1) in the proposition.

Take any $E \,\in\, \mathcal{M}_\xi$ such that $\phi_L(E) \,=\, E$. Fix a holomorphic isomorphism of vector bundles
$$f \,:\, E \,\longrightarrow\, E \bigotimes L\,. \tag{2.8}$$
For each $i \,\in\, \{1\,,2\}$, we have
$$\mathrm{trace}(f^i) \,\in\, H^0(X,\, L^{\otimes i})$$
(see [BNR, § 3], [Hi]). Also, note that $H^0(X,\, L) \,=\, 0$ because L is nontrivial. Therefore, the spectral curve for the pair $(E\,, f)$ is the covering Y_L in (2.2).

There is a holomorphic line bundle η over Y_L such that $\gamma_{L*}\eta$ is isomorphic to E [BNR, § 3], [Hi], where γ_L is the map in (2.2). This proves statement (2) in the proposition.

To prove statement (3), take η_1 and η_2 as in that statement of the proposition. If $\eta_1 \,=\, \tau^*\eta_2$ for some $\tau \,\in\, \mathrm{Gal}(\gamma_L)$, then clearly $\gamma_{L*}\eta_1$ is isomorphic to $\gamma_{L*}\eta_2$.

Now assume that $\gamma_{L*}\eta_1$ is isomorphic to $\gamma_{L*}\eta_2$. Fix an isomorphism
$$\alpha \,:\, E_1 \,:=\, \gamma_{L*}\eta_1 \,\longrightarrow\, \gamma_{L*}\eta_2 \,:=\, E_2\,. \tag{2.9}$$
We now note that
$$\gamma_{L*}\left(\bigoplus_{\tau\in\mathrm{Gal}(\gamma_L)} \eta_1^* \bigotimes \tau^*\eta_2\right) \,=\, \bigoplus_{\tau\in\mathrm{Gal}(\gamma_L)} \gamma_{L*}(\eta_1^* \bigotimes \tau^*\eta_2) \,=\, E_1^* \bigotimes E_2\,. \tag{2.10}$$

Since γ_L is a finite morphism, for any holomorphic vector bundle W on Y_L,
$$H^i(Y_L,\, W) \,=\, H^i(X,\, \gamma_{L*}W) \tag{2.11}$$
for all i. Therefore, from (2.10),
$$H^0(X,\, \mathcal{H}om(E_1\,, E_2)) \,=\, H^0(Y_L,\, \mathcal{H}om(\eta_1\,, \eta_2)) \bigoplus H^0(Y_L,\, \mathcal{H}om(\eta_1\,, \sigma^*\eta_2))\,, \tag{2.12}$$

where σ is the automorphism in (2.5). Consequently, the nonzero element

$$\alpha \in H^0(X, \mathcal{H}om(E_1, E_2))$$

in (2.9) gives a nonzero element in the right–hand side of (2.12). Hence we conclude that either η_1 is isomorphic to η_2 or η_1 is isomorphic to $\sigma^*\eta_2$.

To complete the proof of statement (3) we need to show that η_1 can not be isomorphic to both η_2 and $\sigma^*\eta_2$.

If $\eta_1 = \eta_2 = \sigma^*\eta_2$, then from (2.12) we conclude that

$$\dim H^0(X, \mathcal{H}om(E_1, E_2)) \geq 2\,. \tag{2.13}$$

On the other hand, both E_1 and E_2 are stable vector bundles over X of rank r and degree one (see Lemma 2.1). Hence

$$\dim H^0(X, \mathcal{H}om(E_1, E_2)) \leq 1\,.$$

But this contradicts (2.13). Therefore, $\eta_2 \neq \sigma^*\eta_2$. This completes the proof of the proposition. □

3. Tangential action at fixed points

The holomorphic tangent bundle of $\mathcal{M}_\xi$ will be denoted by $T\mathcal{M}_\xi$.

Let L be any nontrivial line bundle over X of order two. Take any stable vector bundle $E \in \mathcal{M}_\xi$ such that $\phi_L(E) = E$, where ϕ_L is constructed in (1.2). The following lemma describes the spectral decomposition of the differential

$$d\phi_L(E) : T_E\mathcal{M}_\xi \longrightarrow T_E\mathcal{M}_\xi \tag{3.1}$$

at the point $E \in \mathcal{M}_\xi$; here $T_E\mathcal{M}_\xi$ is the fiber of $T\mathcal{M}_\xi$ at E.

Lemma 3.1. *The eigenvalues of the differential $d\phi_L(E)$ in (3.1) are ± 1. The multiplicity of the eigenvalue 1 is $g-1$. The multiplicity of the eigenvalue -1 is $2(g-1)$.*

Proof. Since $\phi_L \circ \phi_L = \mathrm{Id}_{\mathcal{M}_\xi}$, the only possible eigenvalues of $d\phi_L(E)$ are -1 and 1.

Proposition 2.2(2) says that there is a holomorphic line bundle η on Y_L such that

$$E_\eta := \gamma_{L*}\eta \cong E\,.$$

Consider the isomorphism $\gamma_{L*}h$ constructed in (2.7). For any vector bundle W over X, the endomorphism bundle $\mathcal{E}nd(W \bigotimes L) = (W \bigotimes L) \bigotimes (W \bigotimes L)^*$ is canonically identified with $\mathcal{E}nd(W) = W \bigotimes W^*$. Hence the isomorphism $\gamma_{L*}h$ of $\gamma_{L*}\eta$ with $(\gamma_{L*}\eta) \bigotimes L$ defines an automorphism of the vector bundle $\mathcal{E}nd(\gamma_{L*}\eta)$

$$\theta : \mathcal{E}nd(\gamma_{L*}\eta) \longrightarrow \mathcal{E}nd(\gamma_{L*}\eta)\,. \tag{3.2}$$

Let

$$\mathrm{ad}(E_\eta) = \mathrm{ad}(\gamma_{L*}\eta) \subset \mathcal{E}nd(\gamma_{L*}\eta) \tag{3.3}$$

be the subbundle of corank one given by the sheaf of endomorphisms of E_η of trace zero. It is easy to see that θ in (3.2) preserves this subbundle $\mathrm{ad}(\gamma_{L*}\eta)$. Hence θ induces an automorphism

$$\theta_0 : \mathrm{ad}(\gamma_{L*}\eta) \longrightarrow \mathrm{ad}(\gamma_{L*}\eta) \tag{3.4}$$

of the vector bundle $\text{ad}(\gamma_{L*}\eta)$. Let

$$\overline{\theta}_0 \,:\, H^1(X,\, \text{ad}(\gamma_{L*}\eta)) \,\longrightarrow\, H^1(X,\, \text{ad}(\gamma_{L*}\eta)) \tag{3.5}$$

be the automorphism induced by θ_0 in (3.4).

The tangent space $T_E\mathcal{M}_\xi$ is identified with $H^1(X,\, \text{ad}(\gamma_{L*}\eta))$. The differential $d\phi_L(E)$ in (3.1) coincides with the automorphism $\overline{\theta}_0$ constructed in (3.5).

From (2.10) we know that

$$\gamma_{L*}\left((\eta^*\bigotimes\eta)\bigoplus(\eta^*\bigotimes\sigma^*\eta)\right) \,=\, E_\eta^*\bigotimes E_\eta \,=\, \mathcal{E}nd(E_\eta)\,, \tag{3.6}$$

where $E_\eta \,=\, \gamma_{L*}\eta$, and σ is defined in (2.5). From (3.6) and (2.11),

$$H^1(X,\, \mathcal{E}nd(E_\eta)) \,=\, H^1(Y_L,\, \mathcal{H}om(\eta\,,\eta))\bigoplus H^1(Y_L,\, \mathcal{H}om(\eta\,,\sigma^*\eta)) \tag{3.7}$$

(as in (2.12)).

Consider the nontrivial element $\sigma \,\in\, \text{Gal}(\gamma_L) \,=\, C_2$ (see (2.5)). The automorphism θ of $\mathcal{E}nd(\gamma_{L*}\eta)$ in (3.2) preserves the subbundle

$$\gamma_{L*}(\eta^*\bigotimes\sigma^*\eta) \,\subset\, \mathcal{E}nd(E_\eta)$$

in (3.6), and furthermore, θ acts on this subbundle $\gamma_{L*}(\eta^*\bigotimes\sigma^*\eta)$ as multiplication by -1. It is easy see that

$$\gamma_{L*}(\eta^*\bigotimes\sigma^*\eta) \,\subset\, \text{ad}(E_\eta) \,\subset\, \mathcal{E}nd(E_\eta)\,. \tag{3.8}$$

We also note that the automorphism

$$\theta \,\in\, \text{Aut}(\mathcal{E}nd(E_\eta))$$

acts trivially on the subspace

$$\gamma_{L*}(\eta^*\bigotimes\eta) \,\subset\, \mathcal{E}nd(E_\eta)$$

in (3.6). Therefore, the subspace of $H^1(X,\, \text{ad}(\gamma_{L*}\eta))$ on which the automorphism $\overline{\theta}_0$ in (3.5) acts as multiplication by -1 coincides with the subspace

$$H^1(Y_L,\, \mathcal{H}om(\eta\,,\sigma^*\eta)) \,\subset\, H^0(X,\, \text{ad}(E_\eta))$$

in (3.7).

From (2.11) we have $H^i(X,\, \gamma_{L*}(\eta^*\bigotimes\sigma^*\eta)) \,=\, H^i(Y_L,\, \eta^*\bigotimes\sigma^*\eta)$ for all i. From (3.8),

$$H^0(Y_L,\, \eta^*\bigotimes\sigma^*\eta) \,\subset\, H^0(X,\, \text{ad}(E_\eta))\,.$$

But $H^0(X,\, \text{ad}(E_\eta)) \,=\, 0$ because the vector bundle E_η is stable (see Lemma 2.1). Hence

$$H^0(Y_L,\, \eta^*\bigotimes\sigma^*\eta) \,=\, 0\,. \tag{3.9}$$

Since $\text{genus}(Y_L) \,=\, 2g-1$, using Riemann–Roch, from (3.9) it follows that

$$\dim H^1(X,\, \gamma_{L*}(\eta^*\bigotimes\sigma^*\eta)) \,=\, 2(g-1)\,.$$

Therefore, -1 is an eigenvalue of the automorphism θ_0 in (3.4) of multiplicity $2(g-1)$.

We already noted that the only possible eigenvalues of $d\phi_L(E)$ are -1 and 1. Hence 1 is an eigenvalue of the automorphism θ_0 in (3.4) of multiplicity $g-1$. This completes the proof of the lemma. $\square$

Corollary 3.2. *The degree shift* $\iota(L) = g - 1$ *when* $L \in \Gamma$ *is nontrivial, and* $\iota(L) = 0$ *when* L *is trivial.*

Proof. If the eigenvalues are $\exp(2\pi\sqrt{-1}a_j)$, where $0 \leq a_j < 1$ with multiplicity m_j, then by definition

$$\iota(L) = \sum_j a_j m_j \, .$$

So the corollary follows immediately from Lemma 3.1. □

4. Intersection of fixed point sets

Take any $L \in \Gamma \setminus \{\mathcal{O}_X\}$ (see (1.1)). Consider the covering γ_L in (2.2) associated to L. Since the Galois group for γ_L is $\mathbb{Z}/2\mathbb{Z}$, the covering γ_L defines a surjective homomorphism

$$H_1(X,\, \mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}\, .$$

Such a homomorphism gives a nonzero element in $H^1(X,\, \mathbb{Z}/2\mathbb{Z})$.

Let

$$\omega \, : \, \Gamma \, := \, \mathrm{Pic}^0(X)_2 \longrightarrow H^1(X,\, \mathbb{Z}/2\mathbb{Z}) \tag{4.1}$$

be the homomorphism that sends any L to the cohomology class constructed above from it. This homomorphism ω is in fact an isomorphism. Let

$$\mu \, : \, H^1(X,\, \mathbb{Z}/2\mathbb{Z}) \bigotimes_{\mathbb{Z}/2\mathbb{Z}} H^1(X,\, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(X,\, \mathbb{Z}/2\mathbb{Z}) \, = \, \mathbb{Z}/2\mathbb{Z} \tag{4.2}$$

be the cup product. It is known that the isomorphism ω in (4.1) takes μ to the Weil–pairing on $\mathrm{Pic}^0(X)_2$ (see [Mu1, p. 183] for the definition of Weil–pairing).

Fix two nontrivial holomorphic line bundles L and L' over X of order two such that $L \neq L'$. Let $\mathcal{S}(L)$ and $\mathcal{S}(L')$ be the corresponding subvarieties of $\mathcal{M}_\xi$ parametrizing the fixed point sets of ϕ_L and $\phi_{L'}$ respectively (see (1.4)).

Proposition 4.1. *Let* L *and* L' *be nontrivial line bundles of order two over* X *such that* L *is not isomorphic to* L'. *The variety* $\mathcal{S}(L)$ *does not intersect with* $\mathcal{S}(L')$ *if*

$$\mu(\omega(L) \bigotimes \omega(L')) \, = \, 0\, ,$$

where ω *and* μ *are defined in* (4.1) *and* (4.2) *respectively.*

If

$$\mu(\omega(L) \bigotimes \omega(L')) \, \neq \, 0$$

then $\mathcal{S}(L) \bigcap \mathcal{S}(L')$ *is a finite set of cardinality* 2^{2g-2}.

Proof. Take any vector bundle $E \in \mathcal{S}(L)$. Let

$$\mathrm{ad}(E) \subset \mathcal{E}nd(E) \tag{4.3}$$

be the subbundle of corank one defined by the sheaf of trace zero endomorphisms of E. Since $E \in \mathcal{S}(L)$, the vector bundle $E \bigotimes L$ is holomorphically isomorphic to E. Fix a holomorphic isomorphism

$$A \, : \, E \bigotimes L \longrightarrow E\, .$$

This isomorphism A defines a holomorphic homomorphism

$$\varpi : L \longrightarrow \mathcal{E}nd(E) \tag{4.4}$$

of coherent sheaves. Now consider the composition

$$L \xrightarrow{\varpi} \mathcal{E}nd(E) \xrightarrow{\text{trace}} \mathcal{O}_X .$$

Since L is a nontrivial line bundle of degree zero, there is no nonzero holomorphic homomorphism from L to $\mathcal{O}_X$. Hence the above composition of homomorphisms vanishes identically. Therefore, we conclude that the homomorphism ϖ in (4.4) makes L a coherent subsheaf of $\mathrm{ad}(E)$ defined in (4.3).

Take any $E \in \mathcal{S}(L)\bigcap\mathcal{S}(L')$. Given isomorphisms $E \xrightarrow{\alpha} E\bigotimes L$ and $E \xrightarrow{\beta} E\bigotimes L$, we have the composition isomorphism

$$E \xrightarrow{\beta} E\bigotimes L' \xrightarrow{\alpha\otimes \mathrm{Id}_{L'}} E\bigotimes L\bigotimes L' .$$

Consequently,

$$E \in \mathcal{S}(L\bigotimes L') .$$

Therefore, we have an injective homomorphism of coherent sheaves

$$\mathcal{E}(L, L') := L\bigoplus L'\bigoplus(L\bigotimes L') \longrightarrow \mathrm{ad}(E) . \tag{4.5}$$

Since $\mathrm{degree}(\mathcal{E}(L, L')) = \mathrm{degree}(\mathrm{ad}(E))$ (both are zero), this injective homomorphism must be an isomorphism. Therefore, we conclude that

$$\mathcal{E}(L, L') = \mathrm{ad}(E) , \tag{4.6}$$

where $\mathcal{E}(L, L')$ is defined in (4.5).

Fix trivializations of $L\bigotimes L$ and $L'\bigotimes L'$. These two trivializations together give a trivialization of $(L\bigotimes L')^{\otimes 2}$. The three trivializations together give a Lie algebra structure on the fibers of the vector bundle $\mathcal{E}(L, L')$ (see (4.5)) defined by

$$[(a\,,b\,,c)\,,(a'\,,b'\,,c')] := 2\cdot((b'\otimes c)-(b\otimes c')\,,(a'\otimes c)-(a\otimes c')\,,(a'\otimes b)-(a\otimes b')) . \tag{4.7}$$

Given any holomorphic automorphism T of the vector bundle $\mathcal{E}(L, L')$ over X, we get a new Lie algebra structure on the fibers of $\mathcal{E}(L, L')$ by transporting the earlier Lie algebra structure using T. These Lie algebra structures together define an equivalence class of Lie algebra structures on the fibers of $\mathcal{E}(L, L')$. It is straight forward to check that for each point $x \in X$, the Lie algebra $\mathcal{E}(L, L')_x$ defined above is isomorphic to $\mathrm{sl}(2,\mathbb{C})$; to see this use the basis

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

of $\mathrm{sl}(2,\mathbb{C})$.

Consider the Lie algebra structure of the fibers of $\mathrm{ad}(E)$ constructed using the composition of endomorphisms of E. Since L, L' and $L\bigotimes L'$ are all distinct line bundles, and all are different from the trivial line bundle, it can be shown that the isomorphism in (4.6) takes this Lie algebra structure of the fibers of $\mathrm{ad}(E)$ to the above mentioned equivalence class given by the Lie algebra structure constructed in (4.7).

We also note that the group of all holomorphic automorphisms of the vector bundle $\mathcal{E}(L, L') := L \bigoplus L' \bigoplus (L \bigotimes L')$ coincides with $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ with $\mathbb{C}^*$ acting as automorphisms of each direct summand.

Consider the projective bundle

$$\mathcal{P} \longrightarrow X \tag{4.8}$$

of relative dimension one defined by the projectivized nonzero nilpotent elements in the fibers of $\mathcal{E}(L, L')$. So for each point $x \in X$, the fiber $\mathcal{P}_x$ of $\mathcal{P}$ over x is the projectivization of all elements

$$(a\,, b\,, c) \in \mathcal{E}(L, L')_x$$

such that

$$a^2 - b^2 + c^2 = 0\,.$$

Note that since $a \in L_x$, $b \in (L')_x$ and $c \in (L \bigotimes L')_x$, using the trivializations of $L^{\otimes 2}$, $(L')^{\otimes 2}$ and $(L \bigotimes L')^{\otimes 2}$, we have $a^2\,, b^2\,, c^2 \in \mathbb{C}$.

We noted above that the isomorphism in (4.6) takes the natural Lie algebra structure of the fibers of $\mathrm{ad}(E)$ to the equivalence class given by the Lie algebra structure defined in (4.7). Using this it can be deduced that the projective bundle $\mathbb{P}(E)$ over X is isomorphic to $\mathcal{P}$ constructed in (4.8). Indeed, this follows from the above observation and the fact that for any complex vector space W_0 of dimension two, the space of all projectivized nonzero nilpotent elements in $\mathrm{End}_{\mathbb{C}}(W_0)$ is canonically identified with $\mathbb{P}(W_0)$. The identification sends a nilpotent endomorphism N to the line in W_0 defined by the image of N.

The projective bundle $\mathcal{P}$ defines a holomorphic principal $\mathrm{PGL}(2, \mathbb{C})$–bundle over X. Let $\mathrm{ad}(\mathcal{P})$ be the associated adjoint vector bundle. We recall that $\mathrm{ad}(\mathcal{P})$ is the vector bundle associated to the principal $\mathrm{PGL}(2, \mathbb{C})$–bundle $\mathcal{P}$ for the adjoint action of $\mathrm{PGL}(2, \mathbb{C})$ on its own Lie algebra $\mathrm{sl}(2, \mathbb{C})$. It is easy to see that $\mathrm{ad}(\mathcal{P})$ coincides with the direct image of the relative tangent bundle on the total space of the projective bundle $\mathcal{P}$. Since $\mathcal{P}$ is identified with the projective bundle $\mathbb{P}(E)$, it follows immediately that

$$\mathrm{ad}(\mathcal{P}) = \mathrm{ad}(\mathbb{P}(E)) = \mathcal{E}(L, L')\,, \tag{4.9}$$

where $\mathcal{E}(L, L')$ is the vector bundle defined in (4.5).

Consider the second Stiefel–Whitney class

$$w_2(\mathcal{P}) \in H^2(X,\, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

of the projective bundle $\mathcal{P}$ defined in (4.8). Since $\mathcal{P} = \mathbb{P}(E)$, using (4.6) and (4.5) it follows that $w_2(\mathcal{P})$ coincides with

$$\mu(\omega(L) \bigotimes \omega(L')) \in \mathbb{Z}/2\mathbb{Z}$$

where ω and μ are defined in (4.1) and (4.2) respectively. Therefore, if V is a complex vector bundle of rank two over X such that the projective bundle $\mathbb{P}(V)$ is isomorphic to $\mathcal{P}$, then

$$\mathrm{degree}(V) \equiv \mu(\omega(L) \bigotimes \omega(L')) \pmod{2}\,. \tag{4.10}$$

Since $\mathbb{P}(E)$ over X isomorphic to $\mathcal{P}$, from (4.10) we have

$$1 = \mathrm{degree}(E) \equiv \mu(\omega(L) \bigotimes \omega(L')) \pmod{2}\,. \tag{4.11}$$

If $\mu(\omega(L)\bigotimes\omega(L')) \in \mathbb{Z}/2\mathbb{Z}$ vanishes, the two sides of (4.11) are different. Therefore, we conclude that

$$\mathcal{S}(L)\bigcap\mathcal{S}(L') = \emptyset$$

whenever $\mu(\omega(L)\bigotimes\omega(L')) = 0$.

Now we assume that

$$\mu(\omega(L)\bigotimes\omega(L')) = 1 \in \mathbb{Z}/2\mathbb{Z}\,. \tag{4.12}$$

Define $\mathcal{E}(L, L')$ as in (4.5), and define the Lie algebra structure as in (4.7). Construct the projective bundle $\mathcal{P}$ as in (4.8) from this Lie algebra bundle. We noted earlier that $w_2(\mathcal{P})$ coincides with $\mu(\omega(L)\bigotimes\omega(L'))$. Hence from (4.12) it follows that $w_2(\mathcal{P}) \neq 0$. Consequently, there is a holomorphic vector bundle V over X of rank two and odd degree such that $\mathbb{P}(V) = \mathcal{P}$. Fix a holomorphic line bundle L_0 over X such that

$$L_0^{\otimes 2}\bigotimes\bigwedge^2 V = \xi\,.$$

Therefore,

$$E_0 := V\bigotimes L_0 \tag{4.13}$$

is a holomorphic vector bundle over X of rank two such that $\bigwedge^2 E_0 = \xi$ and $\mathbb{P}(E_0)$ is isomorphic to $\mathcal{P}$.

The isomorphism in (4.9) holds. Therefore, from the fact that $\mathbb{P}(E_0)$ is isomorphic to $\mathcal{P}$, we conclude that $L\bigoplus L'$ is a direct summand of the vector bundle $\mathrm{ad}(E_0)$. Consequently, we have

$$E_0 \in \mathcal{S}(L)\bigcap\mathcal{S}(L')\,.$$

If $E_1 \in \mathcal{S}(L)\bigcap\mathcal{S}(L')$, then we have

$$\mathbb{P}(E_1) = \mathcal{P} = \mathbb{P}(E_0)\,.$$

Hence a vector bundle $E_1 \in \mathcal{M}_\xi$ lies in $\mathcal{S}(L)\bigcap\mathcal{S}(L')$ if and only if

$$E_1 = E_0\bigotimes L_1\,,$$

where $L_1 \in \Gamma$ (see (1.1)), and E_0 is constructed in (4.13).

On the other hand,

$$E_0\bigotimes L = E_0 = E_0\bigotimes L'$$

because $E_0 \in \mathcal{S}(L)\bigcap\mathcal{S}(L')$. It can be shown that for any nontrivial holomorphic line bundle $L'' \in \Gamma$ which is different from the three line bundles L, L' and $L\bigotimes L'$, the vector bundle $E_0\bigotimes L''$ is not isomorphic to E_0. Indeed, if $E_0\bigotimes L''$ is isomorphic to E_0, then from (4.6) we know that L'' is a direct summand of $\mathrm{ad}(E_0) = L\bigoplus L'\bigoplus(L\bigotimes L')$. Hence from the uniqueness of decomposition of a vector bundle (see [At, p. 315, Theorem 3]) it follows immediately that the holomorphic line bundle L'' must be isomorphic to one of L, L' and $L\bigotimes L'$. Therefore, we conclude that $E_0\bigotimes L''$ is not isomorphic to E_0 if L'' is different from the three line bundles L, L' and $L\bigotimes L'$.

Consequently, the intersection $\mathcal{S}(L)\bigcap\mathcal{S}(L')$ is a affine space for the quotient group of Γ obtained by quotienting it with the subgroup generated by L and L'. In particular, we have

$$\#(\mathcal{S}(L)\bigcap\mathcal{S}(L')) = (\#\Gamma)/4 = 2^{2g}/4 = 2^{2(g-1)}\,.$$

This completes the proof of the proposition. $\square$

5. Cohomology groups

5.1. **Cohomology of Gal**$(\gamma_L)\backslash\mathcal{S}(L)/\Gamma$**.** Take any $L \in \Gamma \setminus \{\mathcal{O}_X\}$ (see (1.1)). Since Γ is abelian, it acts on the fixed point set $\mathcal{S}(L)$ defined in (1.4). We will compute the cohomology groups of the quotient space $\mathcal{S}(L)/\Gamma$.

Let W_0 be a $\mathbb{Q}$–vector space of dimension $2(g-1)$. Consider the following action of the group $C_2 = \{\pm 1\}$ on W_0: the element $-1 \in C_2$ acts as multiplication by -1. This action of C_2 on W_0 induces an action of C_2 on the exterior algebra $\bigwedge W_0$.

For any even integer i, define

$$d_g(i) := \dim(\bigwedge^i W_0)^{C_2} = \binom{2g-2}{i}, \tag{5.1}$$

in particular, $d_g(0) = 1$, and for any odd positive integer i, define

$$d_g(i) := 0\,. \tag{5.2}$$

Let

$$\mathrm{Prym}(\gamma_L) \subset \mathrm{Pic}^1(Y_L) \tag{5.3}$$

be the Prym variety parametrizing all line bundles η over Y such that

$$\bigwedge^2 \gamma_{L*}\eta = \xi$$

(see [BNR], [Hi], [Mu2]). It is known that $\mathrm{Prym}(\gamma_L)$ is a complex abelian variety of dimension $g-1$.

Proposition 5.1. *For any positive integer i,*

$$\dim H^i(\mathcal{S}(L)/\Gamma,\, \mathbb{Q}) = d_g(i)\,,$$

where $d_g(i)$ is defined in (5.1) *and* (5.2)*. More precisely, the vector space $H^i(\mathcal{S}(L)/\Gamma,\, \mathbb{Q})$ is identified with $(\bigwedge^i W_0)^{C_2}$, where $W_0 = H^1(\mathrm{Prym}(\gamma_L),\, \mathbb{Q})$.*

Proof. Let

$$p_0 : \mathrm{Prym}(\gamma_L) \longrightarrow \mathcal{M}_\xi \tag{5.4}$$

be the morphism defined by $\eta \longmapsto \gamma_{L*}\eta$ (see Lemma 2.1). Using Proposition 2.2(1) if follows that

$$p_0(\mathrm{Prym}(\gamma_L)) \subset \mathcal{S}(L)\,,$$

where $\mathcal{S}(L)$ is defined in (1.4). From Proposition 2.2(2),

$$p_0(\mathrm{Prym}(\gamma_L)) = \mathcal{S}(L)\,.$$

Using Proposition 2.2(3) we know that $\mathrm{Gal}(\gamma_L)$ acts freely on $\mathrm{Prym}(\gamma_L)$, and

$$\mathcal{S}(L) = \mathrm{Prym}(\gamma_L)/\mathrm{Gal}(\gamma_L)\,. \tag{5.5}$$

We will now explicitly describe the action of Γ on $\mathcal{S}(L)$.

The group Γ (see (1.1)) has the following action on the abelian variety $\text{Prym}(\gamma_L)$ defined in (5.3). Take any line bundle $\zeta \in \Gamma$. For any $\eta \in \text{Prym}(\gamma_L)$, we have

$$\bigwedge^2 \gamma_{L*}(\eta \bigotimes \gamma_L^*\zeta) = (\bigwedge^2 \gamma_{L*}\eta) \bigotimes \zeta^{\otimes 2} = \bigwedge^2 \gamma_{L*}\eta = \xi .$$

Therefore, we have a morphism

$$\phi'(\zeta) : \text{Prym}(\gamma_L) \longrightarrow \text{Prym}(\gamma_L) \tag{5.6}$$

defined by $\eta \longmapsto \eta \bigotimes \gamma_L^*\zeta$. Let

$$\phi' : \Gamma \longrightarrow \text{Aut}(\text{Prym}(\gamma_L)) \tag{5.7}$$

be the homomorphism defined by $\zeta \longrightarrow \phi'(\zeta)$. In other words, ϕ' defines an action of Γ on $\text{Prym}(\gamma_L)$.

The map p_0 in (5.4) clearly commutes with the actions of Γ on $\mathcal{M}_\xi$ and $\text{Prym}(\gamma_L)$ defined by ϕ (see (1.3)) and ϕ' (see (5.7)) respectively. Also, the actions of Γ and $\text{Gal}(\gamma_L)$ (see (5.5)) on $\text{Prym}(\gamma_L)$ commute. Hence

$$\text{Gal}(\gamma_L)\backslash\text{Prym}(\gamma_L)/\Gamma = \mathcal{S}(L)/\Gamma . \tag{5.8}$$

Note that since the group $\text{Gal}(\gamma_L)$ is abelian, any right action of $\text{Gal}(\gamma_L)$ is also a left action of $\text{Gal}(\gamma_L)$.

Consider the action of Γ on $\text{Prym}(\gamma_L)$ constructed in (5.7). In the proof of the first statement in Proposition 2.2 we noted that γ_L^*L has a canonical trivialization. Therefore,

$$\phi'(L) = \text{Id}_{\text{Prym}(\gamma_L)} .$$

Take any $\zeta \in \Gamma \setminus \{L , \mathcal{O}_X\}$. Then $\gamma_L^*\zeta$ is a nontrivial holomorphic line bundle on Y_L. Consequently, the translation $\phi'(\zeta)$ in (5.6) is fixed point free. Using this it follows that the quotient $\text{Prym}(\gamma_L)/\Gamma$ is an abelian variety. In particular, the homomorphism

$$H^i(\text{Prym}(\gamma_L)/\Gamma, \mathbb{Q}) \longrightarrow H^i(\text{Prym}(\gamma_L), \mathbb{Q}) \tag{5.9}$$

induced by the quotient map

$$\text{Prym}(\gamma_L) \longrightarrow \text{Prym}(\gamma_L)/\Gamma \tag{5.10}$$

is an isomorphism for all i.

The quotient map in (5.10) clearly intertwines the actions of $\text{Gal}(\gamma_L)$ on $\text{Prym}(\gamma_L)$ and $\text{Prym}(\gamma_L)/\Gamma$. Hence the isomorphism in (5.9) also intertwines the actions of $\text{Gal}(\gamma_L)$. In view of this, from (5.8) we conclude that

$$H^i(\mathcal{S}(L)/\Gamma, \mathbb{Q}) = H^i(\text{Gal}(\gamma_L)\backslash\text{Prym}(\gamma_L), \mathbb{Q})$$

for all i.

Consider the action of $\text{Gal}(\gamma_L)$ on $\text{Prym}(\gamma_L)$. We have an natural isomorphism

$$H^i(\text{Gal}(\gamma_L)\backslash\text{Prym}(\gamma_L), \mathbb{Q}) = H^i(\text{Prym}(\gamma_L), \mathbb{Q})^\sigma ,$$

where $\sigma \in \text{Gal}(\gamma_L)$ is the nontrivial element (see (2.5)), and

$$H^i(\text{Prym}(\gamma_L), \mathbb{Q})^\sigma \subset H^i(\text{Prym}(\gamma_L), \mathbb{Q})$$

is the subspace fixed pointwise by σ. It can be shown that σ acts on $H^1(\text{Prym}(\gamma_L), \mathbb{Q})$ as multiplication by -1. Note that for the action of σ on $H^1(\text{Pic}^1(Y_L), \mathbb{Q})$, the invariant subspace $H^1(\text{Pic}^1(Y_L), \mathbb{Q})^\sigma$ is identified with $H^1(\text{Pic}^1(X), \mathbb{Q})$; here $H^1(\text{Pic}^1(X), \mathbb{Q})$ is

considered as a subspace of $H^1(\mathrm{Pic}^1(Y_L),\,\mathbb{Q})$ using the homomorphism defined by $c \longmapsto \widetilde{\gamma}_L^* c$, where

$$\widetilde{\gamma}_L\,:\,\mathrm{Pic}^1(X)\,\longrightarrow\,\mathrm{Pic}^1(Y_L)$$

is the homomorphism defined by $\zeta \longmapsto \gamma_L^*\zeta$. The natural decomposition

$$H^1(\mathrm{Pic}^1(Y_L),\,\mathbb{Q})\,=\,H^1(\mathrm{Pic}^1(X),\,\mathbb{Q})\bigoplus H^1(\mathrm{Prym}(\gamma_L),\,\mathbb{Q})\,,$$

is preserved by the action of σ, and it acts on $H^1(\mathrm{Pic}^1(X),\,\mathbb{Q})$ and $H^1(\mathrm{Prym}(\gamma_L),\,\mathbb{Q})$ as multiplication by 1 and -1 respectively.

Therefore, σ acts on

$$H^i(\mathrm{Prym}(\gamma_L),\,\mathbb{Q})\,=\,\bigwedge^i H^1(\mathrm{Prym}(\gamma_L),\,\mathbb{Q})$$

as multiplication by $(-1)^i$. Since $\mathrm{Prym}(\gamma_L)$ is an abelian variety of dimension $g-1$, we have $\dim H^1(\mathrm{Prym}(\gamma_L),\,\mathbb{Q})\,=\,2(g-1)$. This completes the proof of the proposition. ☐

For any $i\,\geq\,0$, denote the $\mathbb{Q}$–vector space $H^{i+2\iota(L)}(\mathcal{S}(L)/\Gamma,\mathbb{Q})$ by $A^i(L)$, where $\iota(L)$ is the degree shift. We recall that $\iota(L)\,=\,g-1$ if L is nontrivial, and $\iota(\mathcal{O}_X)\,=\,0$ (see Corollary 3.2). For any nontrivial $L\,\in\,\Gamma$, from Proposition 5.1 we know that $A^*(L)$ is a graded vector space over $\mathbb{Q}$ with $d_g(i)$ generators of degree $i+2(g-1)$. Note that $A^*(\mathcal{O}_X)\,=\,H^*(\mathcal{M}_\xi/\Gamma,\mathbb{Q})$. We get the following description of the Chen–Ruan cohomology group (compare with (1.5)):

$$H^*_{CR}(\mathcal{M}_\xi/\Gamma,\,\mathbb{Q})\,=\,\bigoplus_{L\in\Gamma} A^*(L). \tag{5.11}$$

5.2. **Cohomology of $\mathcal{M}_\xi/\Gamma$.** Consider the action of Γ on $\mathcal{M}_\xi$ given by the homomorphism ϕ in (1.3). It is known that the corresponding action on $H^*(\mathcal{M}_\xi,\,\mathbb{Q})$ of Γ is the trivial action [HN, p. 215, Theorem 1], [AB, p. 578, Proposition 9.7]. Therefore, the homomorphism

$$\psi^*\,:\,H^*(\mathcal{M}_\xi/\Gamma,\,\mathbb{Q})\,\longrightarrow\,H^*(\mathcal{M}_\xi,\,\mathbb{Q}) \tag{5.12}$$

induced by the quotient map

$$\psi\,:\,\mathcal{M}_\xi\,\longrightarrow\,\mathcal{M}_\xi/\Gamma \tag{5.13}$$

is an isomorphism.

There is a holomorphic universal vector bundle $\widetilde{\mathcal{E}}\,\longrightarrow\,X\times\mathcal{M}_\xi$. It is universal in the sense that for each point $m\,\in\,\mathcal{M}_\xi$, the holomorphic vector bundle over X obtained by restricting $\widetilde{\mathcal{E}}$ to $X\times\{m\}$ is in the isomorphism defined by the point m of the moduli space. Any two universal vector bundles over $X\times\mathcal{M}_\xi$ differ by tensoring with a line bundle pulled back from $\mathcal{M}_\xi$. Therefore, the vector bundle

$$\mathcal{U}\,:=\,\mathrm{ad}(\widetilde{\mathcal{E}})\,\subset\,\widetilde{\mathcal{E}}\bigotimes\widetilde{\mathcal{E}}^* \tag{5.14}$$

defined by the sheaf of trace zero endomorphisms is unique up to an isomorphism.

Consider

$$c_2(\mathcal{U})\,\in\,H^4(X\times\mathcal{M}_\xi,\,\mathbb{Q})\,, \tag{5.15}$$

where $\mathcal{U}$ is defined in (5.14). Using Künneth decomposition,

$$c_2(\mathcal{U}) \in \bigoplus_{i=0}^{2} H^i(X, \mathbb{Q}) \bigotimes H^{4-i}(\mathcal{M}_\xi, \mathbb{Q}) = \bigoplus_{i=0}^{2} H_i(X, \mathbb{Q})^* \bigotimes H^{4-i}(\mathcal{M}_\xi, \mathbb{Q}) .$$

Therefore, $c_2(\mathcal{U})$ gives a $\mathbb{Q}$–linear homomorphism

$$H : \bigoplus_{i=0}^{2} H_i(X, \mathbb{Q}) \longrightarrow \bigoplus_{i=0}^{2} H^{4-i}(\mathcal{M}_\xi, \mathbb{Q}) \tag{5.16}$$

such that $H(H_i(X, \mathbb{Q})) \subset H^{4-i}(\mathcal{M}_\xi, \mathbb{Q})$.

It is known that the image of the homomorphism H in (5.16) generates the entire cohomology algebra $\bigoplus_{i>0} H^i(\mathcal{M}_\xi, \mathbb{Q})$ [Ne, p. 338, Theorem 1] (see also [AB, p. 581, Theorem 9.11]).

We noted earlier that the homomorphism ψ^* in (5.12) is an isomorphism. Let

$$\widetilde{H} := (\psi^*)^{-1} \circ H : \bigoplus_{i=0}^{2} H_i(X, \mathbb{Q}) \longrightarrow \bigoplus_{i=0}^{2} H^{4-i}(\mathcal{M}_\xi/\Gamma, \mathbb{Q}) \tag{5.17}$$

be the composition homomorphism. Therefore, the image of $\widetilde{H}$ generates the cohomology algebra of $\mathcal{M}_\xi/\Gamma$.

6. The Chen–Ruan cohomology ring

Take any nontrivial line bundle $L \in \Gamma \setminus \{\mathcal{O}_X\}$ (see (1.1)). Let

$$f : \mathcal{S}(L)/\Gamma \longrightarrow \mathcal{M}_\xi/\Gamma \tag{6.1}$$

be the inclusion map. Let

$$f^* : H^*(\mathcal{M}_\xi/\Gamma, \mathbb{Q}) \longrightarrow H^*(\mathcal{S}(L)/\Gamma, \mathbb{Q}) \tag{6.2}$$

be the pull back operation by the map f in (6.1).

Let

$$p_1 : \mathrm{Prym}(\gamma_L) \longrightarrow \mathcal{S}(L) \tag{6.3}$$

be the quotient map (see (5.5)). Note that the homomorphism of cohomologies with coefficients in $\mathbb{Q}$ induced by p_1 is injective. In fact the pullback operation by p_1 identifies $H^i(\mathcal{S}(L), \mathbb{Q})$ with the invariant part $H^i(\mathrm{Prym}(\gamma_L), \mathbb{Q})^{\mathrm{Gal}(\gamma_L)}$ for all i. Let

$$\iota_0 : \mathrm{Prym}(\gamma_L) \hookrightarrow \mathrm{Pic}^1(Y_L) \tag{6.4}$$

be the inclusion map (see (5.3)). There is a canonical polarization

$$\Theta \in H^2(\mathrm{Pic}^1(Y_L), \mathbb{Q})$$

constructed using the cup product on $H^1(Y_L, \mathbb{Q})$ and the orientation of Y_L.

Proposition 6.1. *Consider the composition $f^* \circ \widetilde{H}$, where $\widetilde{H}$ and f are constructed in* (5.17) *and* (6.2) *respectively. Then*

$$(f^* \circ \widetilde{H})(H_1(X, \mathbb{Q})) = 0 = (f^* \circ \widetilde{H})(H_0(X, \mathbb{Q})) .$$

Furthermore,

$$p_1^*((f^* \circ \widetilde{H})([X])) = 2\iota_0^*\Theta ,$$

where $[X] \in H_2(X, \mathbb{Z})$ *is the oriented generator and* $\Theta \in H^2(\mathrm{Pic}^1(Y_L), \mathbb{Q})$ *is the canonical polarization; the maps* p_1 *and* ι_0 *are constructed in* (6.3) *and* (6.4) *respectively.*

Proof. Consider the map p_0 constructed in (5.4). For any $i \geq 0$, let

$$\widetilde{p}_i \,:\, H^i(\mathcal{M}_\xi\, \mathbb{Q}) \longrightarrow H^i(\mathrm{Prym}(\gamma_L), \mathbb{Q}) \tag{6.5}$$

be the homomorphism defined by $c \longmapsto p_0^* c$. We noted that the homomorphism ψ^* in (5.12) is an isomorphism. We also observed that the homomorphism in (5.9) is an isomorphism. Therefore, to prove the proposition it is enough to show that the following three are valid:

$$\widetilde{p}_3(H(H_1(X, \mathbb{Q}))) \,=\, 0\,, \tag{6.6}$$

$$\widetilde{p}_4(H(H_0(X, \mathbb{Q}))) \,=\, 0\,, \tag{6.7}$$

and

$$\widetilde{p}_2(H([X])) \,=\, 2\iota_0^*\Theta\,, \tag{6.8}$$

where H is the homomorphism in (5.16), and $\widetilde{p}_i$ is constructed in (6.5) (the map ι_0 is defined in (6.4)).

Fix a universal (Poincaré) line bundle

$$\mathcal{L}_0 \longrightarrow Y_L \times \mathrm{Pic}^1(Y_L)\,, \tag{6.9}$$

where Y_L is the covering in (2.2). Let

$$\mathcal{L} \,:=\, (\mathrm{Id}_{Y_L} \times \iota_0)^*\mathcal{L}_0 \longrightarrow Y_L \times \mathrm{Prym}(\gamma_L) \tag{6.10}$$

be the line bundle, where ι_0 in the inclusion map in (6.4).

Consider the vector bundle

$$(\gamma_L \times p_1)^*\mathcal{U} \longrightarrow Y_L \times \mathrm{Prym}(\gamma_L)\,,$$

where $\mathcal{U}$ is the vector bundle in (5.14), and p_1 is the map in (6.3) (recall that $\mathcal{S}(L) \subset \mathcal{M}_\xi$). It is straight forward to check that

(6.11)

$$(\gamma_L \times p_1)^*\mathcal{U} \,=\, (\mathcal{L}^* \bigotimes (\sigma \times \mathrm{Id}_{\mathrm{Prym}(\gamma_L)})^*\mathcal{L}) \bigoplus (\mathcal{L} \bigotimes (\sigma \times \mathrm{Id}_{\mathrm{Prym}(\gamma_L)})^*\mathcal{L}^*) \bigoplus \mathcal{O}_{Y_L \times \mathrm{Prym}(\gamma_L)}\,,$$

where $\mathcal{L}$ is the line bundle in (6.10), and σ is the automorphism in (2.5). From (6.11),

$$(\gamma_L \times p_1)^* c_2(\mathcal{U}) \,=\, -((\sigma \times \mathrm{Id}_{\mathrm{Prym}(\gamma_L)})^* c_1(\mathcal{L}) - c_1(\mathcal{L}))^2 \tag{6.12}$$

Consider the Abel–Jacobi map $Y_L \longrightarrow \mathrm{Pic}^1(Y_L)$ defined by $y \longmapsto \mathcal{O}_{Y_L}(y)$. The corresponding homomorphism

$$\widetilde{B} \,:\, H^1(\mathrm{Pic}^1(Y_L), \mathbb{Q}) \longrightarrow H^1(Y_L, \mathbb{Q}) \,=\, H^1(Y_L, \mathbb{Q})^* \tag{6.13}$$

is an isomorphism; the identification of $H^1(Y_L, \mathbb{Q})$ with $H^1(Y_L, \mathbb{Q})^*$ is given by the cup product on $H^1(Y_L, \mathbb{Q})$. Let

$$B \in H^1(Y_L, \mathbb{Q}) \bigotimes H^1(\mathrm{Pic}^1(Y_L), \mathbb{Q}) \subset H^2(Y_L \times \mathrm{Pic}^1(Y_L), \mathbb{Q}) \tag{6.14}$$

be the element given by the isomorphism $\widetilde{B}$ in (6.13).

The Poincaré line bundle $\mathcal{L}_0$ in (6.9) can be so normalized that

$$c_1(\mathcal{L}_0) \,=\, p_{Y_L}^*([Y_L]) + B\,,$$

where

- $p_{Y_L} \,:\, Y_L \times \mathrm{Pic}^1(Y_L) \longrightarrow Y_L$ is the projection, and $[Y_L] \in H^2(Y_L,\, \mathbb{Z})$ is the oriented generator, and
- B is the cohomology class in (6.14)

(see [ACGH, Ch. 1, § 5] and [ACGH, Ch. IV, § 2]).

Using the above description of $c_1(\mathcal{L}_0)$ together with (6.12) we conclude that (6.7) holds (recall that ψ^* in (5.12) is an isomorphism).

The involution σ in (2.5) defines actions of $\mathbb{Z}/2\mathbb{Z}$ on $H^1(Y_L,\, \mathbb{Q})$ and $\mathrm{Pic}^1(Y_L)$. The action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Pic}^1(Y_L)$ induces an action of $\mathbb{Z}/2\mathbb{Z}$ on $H^1(\mathrm{Pic}^1(Y_L),\, \mathbb{Q})$. The homomorphism $\widetilde{B}$ in(6.13) intertwines the actions of $\mathbb{Z}/2\mathbb{Z}$ on $H_1(Y_L,\, \mathbb{Q})$ and $H^1(\mathrm{Pic}^1(Y_L),\, \mathbb{Q})$. We also note that

$$H^1(Y_L,\, \mathbb{Q})^\sigma \,=\, H^1(X,\, \mathbb{Q})\,, \tag{6.15}$$

and the subspace $H^1(\mathrm{Prym}(\gamma_L),\, \mathbb{Q}) \,\subset\, H^1(\mathrm{Pic}^1(Y_L),\, \mathbb{Q})$ coincides with the subspace on which the nonzero element in $\mathbb{Z}/2\mathbb{Z}$ acts as multiplication by -1 (this was also noted in the proof of Proposition 5.1).

Since $\widetilde{B}$ in(6.13) intertwines the actions of $\mathbb{Z}/2\mathbb{Z}$, it sends the invariant subspace $H^1(\mathrm{Pic}^1(Y_L),\, \mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}}$ to the subspace in (6.15). Using this and (6.12) we now conclude that (6.6) holds.

To prove (6.8), we will first recall a description of the cohomology class

$$H([X]) \,\in\, H^2(\mathcal{M}_\xi,\, \mathbb{Q})\,,$$

where H is constructed in (5.16).

Let $\widetilde{\mathcal{E}}$ be a universal vector bundle over $X \times \mathcal{M}_\xi$ (see (5.14)). Let

$$p_M \,:\, X \times \mathcal{M}_\xi \,\longrightarrow\, \mathcal{M}_\xi \tag{6.16}$$

be the projection. Define the line bundle

$$\mathrm{Det}(\widetilde{\mathcal{E}}) \,:=\, (\bigwedge\nolimits^{\mathrm{top}} R^0 p_{M*}\widetilde{\mathcal{E}})^* \bigotimes (\bigwedge\nolimits^{\mathrm{top}} R^1 p_{M*}\widetilde{\mathcal{E}}) \,\longrightarrow\, \mathcal{M}_\xi\,.$$

Fix a point $x_0 \,\in\, X$. Let

$$\widetilde{\mathcal{E}}_{x_0} \,:=\, \widetilde{\mathcal{E}}\vert_{\{x_0\}\times\mathcal{M}_\xi} \,\longrightarrow\, \mathcal{M}_\xi$$

be the vector bundle over $\mathcal{M}_\xi$. Now define the line bundle

$$\Theta_M \,:\, \mathrm{Det}(\widetilde{\mathcal{E}})^{\otimes 2} \bigotimes (\bigwedge\nolimits^2 \widetilde{\mathcal{E}}_{x_0})^{\otimes(3-2g)} \,\longrightarrow\, \mathcal{M}_\xi\,.$$

Both $\mathrm{Det}(\widetilde{\mathcal{E}})$ and $\bigwedge^2 \widetilde{\mathcal{E}}_{x_0}$ depend on the choice of $\widetilde{\mathcal{E}}$, but Θ_M is independent of the choices of $\widetilde{\mathcal{E}}$ and x_0. In fact, the line bundle Θ_M is the ample generator of $\mathrm{Pic}(\mathcal{M}_\xi) \,\cong\, \mathbb{Z}$.

Since $T\mathcal{M}_\xi \,=\, R^1 p_{M*}\mathcal{U}$, where p_M is the projection in (6.16), from the Hirzebruch–Riemann–Roch theorem it follows that

$$H([X]) \,=\, c_1(T\mathcal{M}_\xi)$$

(note that $R^1 p_{\mathcal{M}*}\mathcal{U} \,=\, 0$). Hence we have

$$H([X]) \,=\, 2\cdot c_1(\Theta_M)\,, \tag{6.17}$$

where H is constructed in (5.16) (see [Ra, p. 69, Theorem 1] and [Ne, p. 338, (1)]).

We will now recall a similar description of the cohomology class Θ on $\mathrm{Pic}^1(Y_L)$.

Take a Poincaré line bundle $\mathcal{L}_0$ on $Y_L \times \mathrm{Pic}^1(Y_L)$ (see (6.9)). Let P_J denote the projection of $Y_L \times \mathrm{Pic}^1(Y_L)$ to $\mathrm{Pic}^1(Y_L)$. Let

$$\mathcal{L}_{x_0} := \mathcal{L}_0|_{\{x_0\}\times \mathrm{Pic}^1(Y_L)} \longrightarrow \mathrm{Pic}^1(Y_L) \tag{6.18}$$

be the line bundle, where x_0 as before is a fixed point of X. Now define the line bundle

$$\Theta_J := \mathrm{Det}(\mathcal{L}_0) \bigotimes \mathcal{L}_{x_0}^{2-g} = (\bigwedge^{\mathrm{top}} R^0 p_{J*}\mathcal{L}_0)^* \bigotimes (\bigwedge^{\mathrm{top}} R^1 p_{J*}\mathcal{L}_0) \bigotimes \mathcal{L}_{x_0}^{2-g} \longrightarrow \mathrm{Pic}^1(Y_L)\,.$$

This line bundle Θ_J does not depend on the choice of $\mathcal{L}_0$, but it depends on the choice of x_0. Since X is connected,

$$c_1(\Theta_J) \in H^2(\mathrm{Pic}^1(Y_L), \mathbb{Q})$$

is independent of x_0. It is known that

$$c_1(\Theta_J) = \Theta\,. \tag{6.19}$$

Let $\mathcal{F}_0 := (\gamma_L \times \mathrm{Id}_{\mathrm{Pic}^1(Y_L)})_*\mathcal{L}_0 \longrightarrow X \times \mathrm{Pic}^1(Y_L)$ be the vector bundle. Using (2.11) we have an isomorphism of line bundles

$$\mathrm{Det}(\mathcal{L}_0) = \mathrm{Det}(\mathcal{F}_0) := (\bigwedge^{\mathrm{top}} R^0 q_{X*}\mathcal{F}_0)^* \bigotimes (\bigwedge^{\mathrm{top}} R^1 q_{X*}\mathcal{F}_0) \longrightarrow \mathrm{Pic}^1(Y_L)\,,$$

where q_X is the projection of $X \times \mathrm{Pic}^1(Y_L)$ to $\mathrm{Pic}^1(Y_L)$. We may choose $\mathcal{L}_0$ and x_0 such that the line bundle $\mathcal{L}_{x_0}$ (see (6.18)) is trivial. Hence comparing (6.17) and (6.19) we conclude that (6.8) holds. This completes the proof of the proposition. $\square$

Recall that we denoted $H^{*+2\iota(L)}(\mathcal{S}(L)/\Gamma, \mathbb{Q})$ by $A^*(L)$, which is a graded vector space over $\mathbb{Q}$ with $d_g(i)$ generators of degree $i + 2(g-1)$ for every nontrivial $L \in \Gamma$, and $A^*(\mathcal{O}_X) = H^*(\mathcal{M}_\xi/\Gamma, \mathbb{Q})$. There is a nondegenerate bilinear Poincaré pairing $\langle\, , \rangle$ for Chen–Ruan cohomology. For $\alpha \in A^*(L)$ and $\beta \in A^*(L')$, the pairing $\langle \alpha\,, \beta\rangle$ is nonzero only when $L' = L^{-1} = L$. In this case it is defined by

$$\langle \alpha\,, \beta\rangle = \int^{\mathrm{orb}}_{\mathcal{S}(L)/\Gamma} \alpha \bigwedge \beta \tag{6.20}$$

Here, and henceforth, we use $\bigwedge$ to represent ordinary cup product. The integral notation $\int^{\mathrm{orb}}_{\mathcal{Y}/G}$ refers to a multiple of the evaluation on the fundamental class of $\mathcal{Y}/G$. This multiple is the reciprocal of the cardinality of the subgroup of G that acts trivially on the manifold $\mathcal{Y}$ (see page 6 of [CR1]). We avoid differential forms unlike [CR1] since we have the coefficients to be $\mathbb{Q}$.

For $\alpha_1 \in A^p(L_1)$, $\alpha_2 \in A^q(L_2)$, the Chen–Ruan product

$$\alpha_1 \bigcup \alpha_2 \in A^{p+q}(L_1 \bigotimes L_2)$$

is defined via the relation

$$\langle \alpha_1 \bigcup \alpha_2\,, \alpha_3\rangle = \int^{\mathrm{orb}}_{\mathbf{S}/\Gamma} e_1^*\alpha_1 \bigwedge e_2^*\alpha_2 \bigwedge e_3^*\alpha_3 \bigwedge c_{\mathrm{top}}\mathcal{F} \tag{6.21}$$

for all $\alpha_3 \in A^*(L_3)$ (it is enough to consider $L_3 = L_1 \bigotimes L_2$), where

$$\mathbf{S} := \bigcap_{i=1}^{3} \mathcal{S}(L_i)$$

and $e_i : \mathbf{S}/\Gamma \longrightarrow \mathcal{S}(L_i)/\Gamma$ are the canonical inclusions. Here $\mathcal{F}$ is a complex Γ–bundle over $\mathbf{S}$ (or equivalently an orbifold vector bundle over $\mathbf{S}/\Gamma$) of rank

$$\mathrm{rank}(\mathcal{F}) \,=\, \dim_{\mathbb{Q}} \mathbf{S} - \dim_{\mathbb{Q}} \mathcal{M}_\xi + \sum_{j=1}^{3} \iota(L_j) \tag{6.22}$$

(see the proof of Theorem 4.1.5 in [CR1]). In general, $c_{\mathrm{top}}\mathcal{F}$ (as defined in [CR1]) is $\mathbb{R}$-valued, but we will see below that it is $\mathbb{Q}$-valued in our case.

If $L_1 = L_2 = \mathcal{O}_X$, then the Chen–Ruan product $\alpha_1 \bigcup \alpha_2$ is the ordinary cup product in $H^*(\mathcal{M}_\xi(r)/\Gamma,\, \mathbb{Q})$.

Since $L_3 = L_1 \bigotimes L_2$, we only need to consider the following remaining cases:

a) $L_1 = L_2 = L \neq \mathcal{O}_X, L_3 = \mathcal{O}_X$
b) $L_1 = L \neq \mathcal{O}_X, L_2 = \mathcal{O}_X, L_3 = L$
c) $L_1 = \mathcal{O}_X, L_2 = L \neq \mathcal{O}_X, L_3 = L$
d) $L_1 \neq \mathcal{O}_X, L_2 \neq \mathcal{O}_X, L_1 \neq L_2, L_3 = L_1 \bigotimes L_2$.

Part of the calculations for the first three cases are analogous. In these cases, $\mathbf{S} = \mathcal{S}(L)$, and by Corollary 3.2,

$$\mathrm{rank}(\mathcal{F}) \,=\, (g-1) - 3(g-1) + 2(g-1) \,=\, 0 \tag{6.23}$$

so that (6.21) reduces to

$$\langle \alpha_1 \bigcup \alpha_2 \,, \alpha_3 \rangle \,=\, \int^{\mathrm{orb}}_{\mathcal{S}(L)/\Gamma} e_1^*\alpha_1 \bigwedge e_2^*\alpha_2 \bigwedge e_3^*\alpha_3 \,. \tag{6.24}$$

6.1. **Case** *a)***.** In this case (6.21) becomes

$$\langle \alpha_1 \bigcup \alpha_2 \,, \alpha_3 \rangle \,=\, \int^{\mathrm{orb}}_{\mathcal{S}(L)/\Gamma} \alpha_1 \bigwedge \alpha_2 \bigwedge e_3^*\alpha_3 \,, \tag{6.25}$$

and e_3 coincides with the inclusion map f in (6.1).

Let us define κ to be the cohomology class

$$\kappa \,=\, \widetilde{H}[X] \,\in\, H^2(\mathcal{M}_\xi/\Gamma,\, \mathbb{Q}) \,=\, A^2(\mathcal{O}_X) \tag{6.26}$$

where $\widetilde{H}$ is constructed in (5.17). We have

$$p_1^* q^* f^*(\kappa) \,=\, 2\iota_0^*\Theta \tag{6.27}$$

(see Proposition 6.1), where f^*, p_1 and ι_0 are constructed in (6.2), (6.3) and (6.4) respectively, and $q : \mathcal{S}(L) \longrightarrow \mathcal{S}(L)/\Gamma$ is the quotient map.

From Proposition 6.1 we have $e_3^*\alpha_3 = 0$ unless α_3 is a linear combination of $\{\kappa^m\}_{m=0}^{g-1}$. Since $\alpha_1 \bigwedge \alpha_2 \in H^{p+q-4(g-1)}(\mathcal{S}(L)/\Gamma,\, \mathbb{Q})$, we know that $\langle \alpha_1 \bigcup \alpha_2 \,, \alpha_3\rangle$ is nonzero only if α_3 is a multiple of κ^{m_0} where $m_0 = 3(g-1) - (p+q)/2$. The class $\alpha_1 \bigwedge \alpha_2 \bigwedge f^*\kappa^{m_0}$ is some multiple $c(\alpha_1, \alpha_2)\,\Omega$ of the normalized top degree cohomology class Ω of $\mathcal{S}(L)/\Gamma$ satisfying

$$\int^{\mathrm{orb}}_{\mathcal{S}(L)/\Gamma} \Omega \,=\, 1\,.$$

This constant $c(\alpha_1, \alpha_2)$ can be computed because we know $f^*(\kappa)$ in terms of the generators of $H^*(\mathcal{S}(L)/\Gamma,\, \mathbb{Q})$ (see (6.27)). We obtain

$$\langle \alpha_1 \bigcup \alpha_2\, , \alpha_3 \rangle \;=\; \begin{cases} c(\alpha_1, \alpha_2)\, d & \text{if } \alpha_3 = d\, \kappa^{m_0} \\ 0 & \text{otherwise.} \end{cases} \tag{6.28}$$

In the present case, from (6.20),

$$\int_{\mathcal{M}_\xi/\Gamma} (\alpha_1 \bigcup \alpha_2) \bigwedge \alpha_3 \;=\; \begin{cases} c(\alpha_1, \alpha_2)\, d & \text{if } \alpha_3 = d\, \kappa^{m_0} \\ \\ 0 & \text{otherwise.} \end{cases} \tag{6.29}$$

Hence we obtain

$$\alpha_1 \bigcup \alpha_2 \;=\; \frac{c(\alpha_1, \alpha_2)}{v} \kappa^{m_1}\, , \tag{6.30}$$

where $m_1 = 3(g-1) - m_0 = \frac{p+q}{2}$ and

$$v = \int_{\mathcal{M}_\xi/\Gamma} \kappa^{3(g-1)}\, . \tag{6.31}$$

Consider H constructed in (5.16). Thaddeus calculated that

$$\int_{\mathcal{M}_\xi} H([X])^{3g-3} \;=\; \frac{(3g-3)!}{(2g-2)!} 2^{2g-2}(2^{2g-2} - 2)|B_{2g-2}|\, ,$$

where B_{2g-2} is the Bernoulli number (see [Th, p. 147, (29)] and the line following it). Note that v in (6.31) satisfies the condition

$$v \;=\; \frac{1}{2^{2g}} \int_{\mathcal{M}_\xi} H([X])^{3g-3}\, .$$

6.2. **Case** b)**.** In this case (6.21) becomes

$$\langle \alpha_1 \bigcup \alpha_2\, , \alpha_3 \rangle \;=\; \int^{\text{orb}}_{\mathcal{S}(L)/\Gamma} \alpha_1 \bigwedge e_2^* \alpha_2 \bigwedge \alpha_3\, , \tag{6.32}$$

where e_2 is the inclusion $f : \mathcal{S}(L)/\Gamma \to \mathcal{M}_\xi/\Gamma$. Comparing (6.32) with (6.20) we get

$$\int^{\text{orb}}_{\mathcal{S}(L)/\Gamma} (\alpha_1 \bigcup \alpha_2) \bigwedge \alpha_3 \;=\; \int^{\text{orb}}_{\mathcal{S}(L)/\Gamma} (\alpha_1 \bigwedge f^* \alpha_2) \bigwedge \alpha_3 \tag{6.33}$$

for all α_3. Thus we deduce

$$\alpha_1 \bigcup \alpha_2 \;=\; \alpha_1 \bigwedge f^* \alpha_2 \tag{6.34}$$

Note that $f^* \alpha_2 \;=\; 0$ unless α_2 is scalar multiple of a power of κ.

6.3. **Case** c)**.** By an argument very similar to case b), we get

$$\alpha_1 \bigcup \alpha_2 \;=\; f^* \alpha_1 \bigwedge \alpha_2 \tag{6.35}$$

6.4. **Case** d)**.** We invoke Proposition 4.1. If $\mu(\omega(L_1)\bigotimes\omega(L_2))=0$ then

$$\mathbf{S}\,=\,\mathcal{S}(L_1)\bigcap\mathcal{S}(L_2)\,=\,\emptyset$$

and consequently the Chen–Ruan product

$$\alpha_1\bigcup\alpha_2\,=\,0$$

for all $\alpha_i\in A^*(L_i)$, $i\,=\,1,2$. On the other hand, if $\mu(\omega(L_1)\bigotimes\omega(L_2))\,=\,1$, then $\mathbf{S}/\Gamma$ is a point modulo a finite group of order 4. For dimensional reasons, we have $c_{\text{top}}\mathcal{F}=1$ and

$$\langle\alpha_1\bigcup\alpha_2\,,\alpha_3\rangle\,=\,\begin{cases}\frac{1}{4}\alpha_1\alpha_2\alpha_3 & \text{if } \alpha_i\in A^{2g-2}(L_i)\ \forall\, i\\[2ex] 0 & \text{otherwise.}\end{cases}\tag{6.36}$$

Therefore by (6.20), if Ω' denotes the normalized top degree cohomology class on $\mathcal{S}(L_1\bigotimes L_2)/\Gamma$ such that

$$\int^{\text{orb}}_{\mathcal{S}(L_1\bigotimes L_2)/\Gamma}\Omega'\,=\,1\,,$$

then we have

$$\alpha_1\bigcup\alpha_2\,=\,\begin{cases}\frac{1}{4}\alpha_1\alpha_2\Omega' & \text{if } \alpha_i\in A^{2g-2}(L_i)\ \forall\, i\\[2ex] 0 & \text{otherwise.}\end{cases}\tag{6.37}$$

Acknowledgements. This work was carried out during a visit of the first author to ISI and a visit of the second author to TIFR. We record our thanks to these two institutes.

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